MATH 3060 Tutorial 10

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November 16, 2022

1. A family of equicontinuous functions on a compact metric space (e.g $[0,1]$) is pointwise bounded if and only if it is uniformly bounded.

2. Let (g_n) be a family of function on $\mathbb{R}$ with $\int_0^1 |g|^p < M$ for some positive number M. Define
$$f_n(x) = \int_0^x g_n(t)dt.$$
Show that $\{f_n\}$ is uniformly equicontinuous.

3. Let $\{f_n\}$ be a family of smooth functions on $[-1,1]$ so that
   (a) $f_n^{(i)}(0) = 0$ for each n and $i = 0, 1, 2, 3$.
   (b) $\{f_n^{(3)}\}$ is uniformly bounded.

   Show that $\{f_n\}$ is precompact.

4. Let $\{f_n : \mathbb{C} \to \mathbb{C}\}$ be a sequence of functions, and let U be a bounded open subset of $\mathbb{C}$.
   (a) Assume U is a disc and suppose $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded on U, show that $\{f_n\}$ is precompact.
   (b) Suppose $\{f_n\}$ is uniformly bounded on U and each f_n is holomorphic, show that there exists a subsequence f_{n_k} with converges pointwise to a continuous function, show that the convergence is uniform on every compact subset of U.

5. (Next time) Generalize the Arzela-Ascoli Theorem to the case when
   (a) the codomain is a metric space.
   (b) both the domain and codomain are metric spaces.